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Towards the solution of non-commutative YM_2: Morita equivalence and large-N limit

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Abstract: In this paper we shall investigate the possibility of solving U(1) theories on the non-commutative (NC) plane for arbitrary values of θ by exploiting Morita equivalence. This duality maps the NC U(1) on the two-torus with a rational parameter θ to the standard U(N) theory in the presence of a 't Hooft flux, whose solution is completely known. Thus, assuming a smooth dependence on θ, we are able to construct a series rational approximants of the original theory, which is finally reached by taking the large-N limit at fixed 't Hooft flux. As we shall see, this procedure hides some subletities since the approach of N to infinity is linked to the shrinking of the commutative two-torus to zero-size. The volume of NC torus instead diverges and it provides a natural cut-off for some intermediate steps of our computation. In this limit, we shall compute both the partition function and the correlator of two Wilson lines. A remarkable fact is that the configurations, providing a finite action in this limit, are in correspondence with the non-commutative solitons (fluxons) found independently by Polychronakos and by Gross and Nekrasov, through a direct computation on the plane.

Keywords: Field Theories in Lower Dimensions, $1/N$ Expansion, Duality in Gauge Field Theories, Non-Commutative Geometry.

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Contents

1. Introduction

Non-commutative field theories have gained a central role in the recent developments of string theory. The initial interest was motivated by the results presented in [1], where non-commutative geometry was found to be the natural tool to classify new toroidal compactifications of $M-$theory in the presence of a constant background three-form field. Later on, in [2], non-commutative gauge theories were shown to appear in IIA/B superstring theory in a particular decoupling limit of $D-$branes with a $NS-NS$ two-form background turned on. This possibility to embed consistently a non-commutative field theory into a string theory has stimulated a large amount of studies, trying to understand classical and quantum non-commutative dynamics both at perturbative [3] and at non-perturbative [4] level.

Non-commutative field theories (and in particular gauge theories) present a large variety of new phenomena not completely understood even in the basic cases: at perturbative level the UV/IR mixing[1] [3, 5] complicates the renormalization program (see however [6] for a recent discussion of renormalization in non-commutative QFT) and it seems to produce tachyonic instabilities [7]. At the same time an entirely new family of classical solutions has been discovered [8], and their role in the quantum dynamics is completely unknown (see however [9] where the effects of instantons in $N = 2$ non-commutative SUSY gauge theory were considered). Moreover, gauge theories on the non-commutative torus exhibit a fascinating property, not shared by their commutative ancestors, that goes under the name of Morita equivalence. Roughly speaking, Morita equivalence establishes a relation between gauge theories defined on different non-commutative tori: gauge theories characterized by diverse ranks of the gauge group, flux numbers and non-commutative parameters are seen to be equivalent [10]. This beautiful and absolutely general mathematical property has an elegant and simple interpretation when a string theory embedding of the field theoretical

[1] We give here only the references where the phenomenon was discovered in four-dimensional scalar and gauge theories.

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model is available and the non-commutative torus originates from a compactification procedure. In fact, in this case, Morita equivalence can be viewed as a consequence of the more familiar T-duality [11]. Nevertheless, we must stress again that it holds without any reference to string theory and it is a general (non-perturbative) property, depending just on the geometrical data of the theory itself.

This deep geometrical origin suggests that Morita equivalence must play a central role in understanding some basic facts on non-commutative gauge theories both on T^D and on $\mathbb{R}^D$ (the non-commutative $\mathbb{R}^D$ can be recovered as a suitable large volume limit of a non-commutative torus T^D). To this purpose, a particular promising feature is that, under a Morita transformation, a U(1) gauge theory on a non-commutative torus of *rational* parameter[2] θ is shown to be equivalent to a certain U(N) Yang-Mills theory, in the presence of a ‘t Hooft flux [12], defined on a *commutative* torus. This is not case when θ is irrational (see also [13] for a discussion on the phase structure of "irrational" theories). Although the question of the smooth dependence on θ of the theory is still under investigation [14, 15], this unexpected link opens the concrete possibility to study a non-commutative gauge theory for a generic value of the non-commutative parameter starting from a series of commutative approximants [16]. What we have in mind is, of course, to define a limiting procedure for the commutative theory in order to reach a general non-commutative parameter on the non-commutative euclidean space. We did not attempt to do this in four dimensions: we choose the simpler two-dimensional case, where a complete understanding of the commutative theory is available [17]. At the same time the general solution of non-commutative U(1) theory in $D = 2$ does not exist, while previous studies on classical solutions [18, 19] and Wilson loops [20] show that highly non-trivial aspects are involved. In this paper we propose a limiting procedure to construct U(1) theory on the non-commutative plane, for general θ, starting from a two-torus of rational parameter: it is a particular large-N limit at a fixed value of the 't Hooft flux. This limit was also recently considered in [15, 21], where, however, the four-dimensional case was under study.

In this limit some unexpected, but promising effects emerge. Since the 't Hooft flux is kept fixed, the large volume limit (encoded in the large-N limit) corresponds in the commutative approximants to the shrinking of the torus to zero-size. This make our analysis more involved, because the behavior of YM_2 on a small torus is quite delicate [22]. At the same time, the volume of the non-commutative torus diverges and it becomes the natural cut-off for the intermediate steps of the computations. Moreover, in this procedure, some contributions are naturally singled out. We shall call them *finite action configurations*.[3] It is easy to see that they are in correspondence with the classical solutions carrying a non-trivial flux (fluxons) discussed by Polychronakos [18] and by Gross and Nekrasov [19]. Under the semiclassical assumption that these are the only configurations dominating our limit, we obtain an exact partition function which has the nice property to be extensive. We must recall that, in the commutative case the semiclassical approximation is exact [23] and this may suggest a more general validity of our results.

[2] In the following we shall use a bidimensional language since we shall deal with only this case. In higher dimensions the parameter θ, for example, is substituted by an antisymmetric matrix $\theta_{\mu\nu}$.

[3] The origin of this name will become manifest in section 3.

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The setting for computing correlators of Wilson lines is also presented. The complete computations is carried out in the case of two lines and its semiclassical interpretation in terms of fluxons is given. This technique allows us to tackle also the computations of others observables such as closed Wilson loops and their correlators. This question as well as the one on the exactness of the semiclassical assumption is under investigation at the present [24].

2. Non-commutative U(1) theory on the torus and Morita equivalence

Gauge theories on the non-commutative plane can be constructed by replacing, in the usual Yang-Mills action, the ordinary commutative product of functions with the Moyal $\star$-product. Its definition is given by

$$f(x)\star g(x)=\exp\left(i\frac{\theta^{\mu\nu}}{2}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\right)f(x)g(y)|_{y\to x}\,. \tag{2.1}$$

Notice that, endowed with this product, the commutative coordinates x^μ are promoted to satisfy the familiar Heisenberg algebra

$$x^\mu\star x^\nu-x^\nu\star x^\mu=i\;\theta^{\mu\nu}\,. \tag{2.2}$$

This signals that noncommutativity will, in general, violate Lorentz invariance. However, in two dimensions, which is the case under investigation, the relation (2.2) does not break Lorentz (or euclidean) invariance, since $\theta^{\mu\nu}$ can be always expressed in term of the invariant Levi-Civita tensor,

$$\theta^{\mu\nu}=\theta\;\epsilon^{\mu\nu}\,. \tag{2.3}$$

The action of the non-commutative U(N) Yang-Mills theory can be thus written in this general form

$$S=\frac{1}{4g^2}\int dx^2\,\mathrm{Tr}\left[(F_{\mu\nu}+\Phi_{\mu\nu}\mathbb{I})\star(F^{\mu\nu}+\Phi^{\mu\nu}\mathbb{I})\right], \tag{2.4}$$

where

$$F_{\mu\nu}(x)=\partial_\mu A_\nu-\partial_\nu A_\mu-i(A_\mu\star A_\nu-A_\nu\star A_\mu) \tag{2.5}$$

and $\Phi_{\mu\nu}=\Phi\epsilon_{\mu\nu}$ is a U(1) background term whose meaning will become clear shortly. We notice that a new dimensional parameter, θ, has been introduced into the theory through the star-product, having exactly the same dimensions of the space-time ($[length]^2$), a point that will be appreciate later.

When we compactify both coordinates to go from the plane to the torus, this theories exhibit an SO$(2,2,\mathbb{Z})$ Morita equivalence which is inherited from string theory T-duality [11]. The very same property has also been demonstrated explicitly without recourse to either string theory or supersymmetry [10]. The equivalence connects different non-commutative gauge theories living on different non-commutative tori: in fact the duality group has an SL$(2,\mathbb{Z})$ subgroup which acts as follows on the geometrical and gauge data

$$\begin{pmatrix}m'\\N'\end{pmatrix}=\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}m\\N\end{pmatrix},\qquad\Theta'=\frac{c+d\Theta}{a+b\Theta}, \tag{2.6}$$

$$(R')^2=R^2(a+b\Theta)^2\,,\qquad(g')^2=g^2|a+b\Theta|\,,\qquad\tilde\Phi'=(a+b\Theta)^2\tilde\Phi-b(a+b\Theta)\,, \tag{2.7}$$

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where $\Theta \equiv \theta/(2\pi R^2)$, $\tilde{\Phi} = 2\pi R^2 \Phi$ and $(2\pi R)$ is the circumference of the torus, which for simplicity we take to be square. The first entry, m, in the column-vector denotes the magnetic flux, while N is the rank of the gauge group. The parameter of the transformation are integer numbers, constrained by the equation $ad - bc = 1$. Let us notice that changing the overall sign of the matrix realizing the $\mathrm{SL}(2,\mathbb{Z})$ transformation leads to the same Θ'. The sign has to be chosen so that the new gauge group has positive rank. An important observation, that will be crucial for our development, is that the θ parameter, defining the Moyal product, scales as the area of the torus, while Morita equivalence only involves the adimensional quantities Θ. Another important point is that the background connection Φ transforms inhomogeneously under Morita, therefore it is, in general, non-vanishing along the $\mathrm{SL}(2,\mathbb{Z})$ orbit. Let us now consider a U(1) gauge theory with *rational* $\Theta = -c/N$ (with $\gcd(c,N) = 1$), magnetic flux $m_{\rm nc}$ and, for simplicity, vanishing background connection: a $\mathrm{SL}(2,\mathbb{Z})$ transformation of the form

$$M = \begin{pmatrix} a & b \\ c & N \end{pmatrix}, \tag{2.8}$$

brings the theory to $\Theta = 0$, giving therefore an ordinary but non-abelian theory. In eq. (2.8) we choose the global sign of c and N and consequently that of a and b so that the new rank of the gauge group is positive.[4] Working out the effect of M via eq. (2.6) we see that the new gauge group is

$$N_c = c\, m_{\rm nc} + N\,. \tag{2.9}$$

The magnetic flux is also changed, and we may express the parameter b as a linear combination of the non-commutative magnetic flux and the commutative one

$$b = m_c - a\, m_{\rm nc}\,. \tag{2.10}$$

Here m_c is the magnetic flux (the "commutative" flux). The parameter a and m_c are then constrained by the $\det M = 1$ condition that is

$$a\ N - c(m_c - a\ m_{\rm nc}) = a\ N_c - c\ m_c = 1\,. \tag{2.11}$$

This Diophantine equation determines a modulo c and m_c modulo N_c. An important point in our analysis will be to show that these ambiguities does not affect our limiting procedure. A background connection has also to be introduced into the theory, according to eq. (2.7), leading to

$$\tilde{\Phi}_c = -\frac{b}{N} = -\frac{m_c\ - a\ m_{\rm nc}}{N}\,. \tag{2.12}$$

Our original non-commutative theory is therefore equivalent to an ordinary theory on a torus with area shrunk by a factor N^2

$$R_c^2 = \frac{R_{\rm nc}^2}{N^2}\,. \tag{2.13}$$

[4]The sign ambiguity in M can be in fact simply encoded in the following observation: the couples (c,N) and $(-c,-N)$ give the same Θ. The change sign of the couple (a,b) is then constrained by the Diophantine relation $a\,N - c\,b = 1$. To be precise, this relation determines (a,b) only up to a solution of the homogeneous equation $a\ N - c\ b = 0$, but this does not alter our conclusion.

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The coupling constant has also become weaker of a factor N: $g_c^2 = g_{\rm nc}^2/N$. Thus one could take the point of view that a non-commutative theory exists at least for rational Θ, and then try to define the theory at irrational values by approaching it with an infinite sequence of rational numbers. This possibility has been advocated a certain number of times in the literature [14, 15, 21, 25], but, at least at our knowledge, non-concrete computation have been proposed in two dimensions. We take here a slightly different point of view in that we want to construct the non-commutative theory on the non-commutative plane with arbitrary finite θ parameter: we intend therefore to perform the limit $R_{\rm nc} \to \infty$ (decompactifying the original non-commutative torus where the non-commutative U(1) theory was defined), maintaining finite the dimensional parameter θ.[5] We recall in fact that on the plane θ has dimension $[length]^2$, that is not related to any obvious geometrical quantity. Because we have

$$\theta = 2\pi R_{\rm nc}^2 \Theta = -2\pi R_{\rm nc}^2 \frac{c}{N}\,, \tag{2.14}$$

an easy way to realize our task is to work with a radius

$$R_{\rm nc}^2 = -N\frac{\theta}{2\pi c}\,, \tag{2.15}$$

and taking the limit $N \to \infty$. The value of θ can now be taken completely general, therefore we can freely choose c since this will correspond to a rescaling of the final θ. The Morita equivalent theory with $\theta = 0$ is therefore defined by the following data

$$R_c^2 = -\frac{\theta}{2\pi}\frac{1}{c\;N} = \frac{|\theta|}{2\pi|c\;N|}\,, \tag{2.16}$$

$$g_c^2 = \frac{g_{\rm nc}^2}{N}\,. \tag{2.17}$$

The last equality in (2.16) is a consequence of the fact that θ and Θ are taken to have the same sign because of (2.14). Next we have to satisfy $\mathrm{SL}(2,\mathbb{Z})$ constraint:

$$m_c = \frac{1}{c}(a\;N_c - 1)\,. \tag{2.18}$$

It may seem strange, at this level, that the commutative Chern class is not fully determined (we have the arbitrary integer a in its definition), but we will see how its ambiguity does not affect the final result. Notice that as N goes to infinity, for any fixed non-commutative flux, the Chern class of the equivalent commutative gauge theory has to scale accordingly. Equations (2.16), (2.17) and (2.18) define the limit we would like to perform: it is clear that this is not the usual 't Hooft limit on a commutative torus. Equation (2.17) alone would define the usual large-N limit: in addition eq. (2.16) means that we have also to perform a small area limit and eq. (2.18) forces us to push the first Chern number to infinity. Moreover we have to consider the effect of the U(1) background connection, that will exactly compensate the arbitrariness of a. What we are going to do in the next section, is to use the exact solution of the Yang-Mills theory on the torus [17] in order to construct the Morita equivalent theory for finite N, to implement the presence of the flux and of the background connection, and eventually to perform our large-N limit.

[5] After having completed the computations the very same proposal appears in [15, 21].

3. The partition function of the Morita equivalent theory and its large-N limit

The 't Hooft limit of two-dimensional Yang-Mills theory is a well-studied problem: in particular Gross and Taylor has shown [26] that when the space-time is a compact Riemann surface a string theory emerges in the large-N limit. The dependence of the theory on the size of the area has also been investigated: on the sphere it does exist a critical area separating a strong coupling phase from a weak coupling phase [27]. The phase transition does not occur for genus greater than one: on the torus it would happen just at zero area. The small area behavior on the torus has been studied in [28], in connection with the string description: in our case we have to study a similar limit, with the area scaling as $1/N$. No one, instead, has considered the possibility to perform also a limit on the first Chern class, that is a peculiar element in the Morita equivalent description.

In the following we will consider the case of a U(1) non-commutative theory with vanishing first Chern class ($m_{\rm nc} = 0$) and vanishing background connection ($\Phi_{\rm nc} = 0$). We leave open for future investigations the possibility to sum over the different non-commutative Chern classes.

We start by considering the U(N) theory defined by the following action

$$S = \frac{1}{4g_c^2}\int dx^2\, \mathrm{Tr}\left[\left(F_{\mu\nu} - \frac{m}{2\pi R_c^2 N}\epsilon_{\mu\nu}\mathbb{I}\right)\left(F^{\mu\nu} - \frac{m}{2\pi R_c^2 N}\epsilon^{\mu\nu}\mathbb{I}\right)\right], \tag{3.1}$$

where the explicit form of the background connection, $\Phi_c = -\frac{m}{2\pi R_c^2 N}\mathbb{I}$ has been taken into account. The Chern class of the U(N) field is m

$$m = \frac{1}{4\pi}\int dx^2\, \mathrm{Tr}[F_{\mu\nu}\epsilon^{\mu\nu}]\,. \tag{3.2}$$

It is therefore possible to work out the Φ-dependence in eq. (3.1)

$$S = \frac{1}{4g_c^2}\int dx^2\, \mathrm{Tr}[F_{\mu\nu}F^{\mu\nu}] - \frac{2\pi^2 m^2}{g_c^2 A_c N}\,, \tag{3.3}$$

where $A_c = 4\pi^2 R_c^2$ is the area of the commutative torus. We have now to compute the partition function of a U(N) theory with Chern class equal to m and the additional (constant) term present in eq. (3.3). Let us start with the general Migdal's formula for U(N) (on genus one)

$$\mathcal{Z} = \sum_R \exp\left[-\frac{g_c^2 A_c}{2}C_2(R)\right], \tag{3.4}$$

where $C_2(R)$ is the value of the second Casimir operator in the representation R. The sum runs over the irreducible representation of the gauge group: in the U(N) case the representations R can be labeled by a set of integers $n_i = (n_1, \ldots, n_N)$, related to the Young tableaux, obeying the ordering $+\infty > n_1 > n_2 > \cdots > n_N > -\infty$. In terms of n_i we have for the second Casimir

$$C_2(R) = C_2(n_1, \ldots, n_N) = \frac{N}{12}(N^2-1) + \sum_{i=1}^{N}\left(n_i - \frac{N-1}{2}\right)^2. \tag{3.5}$$

The dependence on the product $g_c^2 A_c$ is peculiar of two-dimensional Yang-Mills theories, that are invariant under area-preserving diffeomorphisms. Using the permutation symmetry we get

$$\mathcal{Z} = \frac{1}{N!} \sum_{n_1 \neq n_2 \neq \cdots \neq n_N} \exp\left[-\frac{g_c^2 A_c}{2} \sum_{i=1}^{N} \left(n_i - \frac{N-1}{2}\right)^2\right]; \tag{3.6}$$

we have disregarded the overall constant term, present in the Casimir, linked to a cosmological constant contribution that plays no dynamical role in this context. To fix the Chern class we factorize the U(1) part: we define

$$n_1 = \lambda\,, \qquad \bar{n}_i = n_i - n_1\,, \qquad i = 2, \ldots, N\,, \tag{3.7}$$

obtaining

$$\begin{aligned}\mathcal{Z} = \frac{1}{N!} \sum_{\lambda=-\infty}^{+\infty} \sum_{\bar{n}_i \neq \bar{n}_j \neq 0} \exp\left[-\frac{g_c^2 A_c N}{2}\left(\lambda - \frac{N-1}{2} + \frac{1}{N}\sum_{i=2}^{N} \bar{n}_i\right)^2\right] \times \\ \times \exp\left[-\frac{g_c^2 A_c}{2}\left(\sum_{i=2}^{N} \bar{n}_i^2 - \frac{1}{N}\left(\sum_{i=2}^{N} \bar{n}_i\right)^2\right)\right].\end{aligned} \tag{3.8}$$

Next we introduce the identity

$$1 = \sum_{l=0}^{N-1} \delta_N\left(l + \frac{N(N-1)}{2} - \sum_{i=2}^{N} \bar{n}_i\right), \tag{3.9}$$

where δ_N is the N-periodic delta function, $\delta_N(x) = \frac{1}{N}\sum_{k=0}^{N-1} \exp[-2\pi i \frac{k}{N} x]$. By using eq. (3.9) we rewrite our partition function as

$$\begin{aligned}\mathcal{Z} = \frac{1}{NN!} \sum_{l,k=0}^{N-1} \sum_{\bar{n}_i \neq \bar{n}_j \neq 0} \exp\left[-\frac{g_c^2 A_c}{2}\left(\sum_{i=2}^{N} \bar{n}_i^2 - \frac{1}{N}\left(\sum_{i=2}^{N} \bar{n}_i\right)^2 - \right.\right. \\ \left.\left. - 2\pi i k\left(\frac{l}{N} + \frac{(N-1)}{2} - \sum_{i=2}^{N} \frac{\bar{n}_i}{N}\right)\right)\right] \times \\ \times \left(\sum_{\lambda=-\infty}^{+\infty} \exp\left[-\frac{g_c^2 A_c N}{2}\left(\lambda + \frac{l}{N}\right)^2\right]\right).\end{aligned} \tag{3.10}$$

In order to single out the contribution of the bundles of Chern class m, we Poisson resum over λ

$$\sum_{\lambda=-\infty}^{+\infty} \exp\left[-\frac{g_c^2 A_c N}{2}\left(\lambda + \frac{l}{N}\right)^2\right] = \sqrt{\frac{2\pi}{g_c^2 A_c N}} \sum_{\lambda=-\infty}^{+\infty} \exp\left[-\frac{2\pi^2}{g_c^2 A_c N}\lambda^2 - 2\pi i \lambda \frac{l}{N}\right]. \tag{3.11}$$

The sum over l can be done giving a $\delta_N(\lambda - k)$, leading to the factorization

$$Z = \sum_{k=0}^{N-1} \left(\sqrt{\frac{2\pi}{g_c^2 A_c N}} \sum_{\lambda = k \mod N} \exp\left[-\frac{2\pi^2}{g_c^2 A_c N}\lambda^2\right] \mathcal{Z}_k\right). \tag{3.12}$$

We have therefore obtained the factorization of the U(N) partition function according to U(N) = U(1) $\otimes$ SU(N)/$\mathbb{Z}_N$: we have that

$$\begin{aligned} \mathcal{Z}_k &= \frac{1}{N!} \sum_{\bar{n}_i \neq \bar{n}_j \neq 0} \exp \left[-\frac{g_c^2 A_c}{2} \left(\sum_{i=2}^N \bar{n}_i^2 - \frac{1}{N} \left(\sum_{i=2}^N \bar{n}_i \right)^2 \right) - \frac{2\pi i k}{N} \left(\frac{N(N-1)}{2} - \sum_{i=2}^N \bar{n}_i \right) \right] \\ &= \sum_R \exp \left[-\frac{g_c^2 A_c}{2} C_2(R) \right] \frac{\mathcal{X}_R(e^{2\pi i \frac{k}{N}})}{d_R} . \end{aligned} \tag{3.13}$$

$\mathcal{Z}_k$ is easily seen to coincide with the SU(N) partition function in the k-'t Hooft sector [12], the sum over the SU(N) irreducible representations R being weighted with the character $\mathcal{X}_R(e^{2\pi i \frac{k}{N}})$ of the k-th N-root of the identity. The partition function of the U(N) theory with first Chern class m is therefore

$$\mathcal{Z}^{(m)}_{\mathrm{U}(N)} = \sqrt{\frac{2\pi}{g_c^2 A_c N}} \exp \left[-\frac{2\pi^2}{g_c^2 A_c N} m^2 \right] \mathcal{Z}_m . \tag{3.14}$$

Coming back to eq. (3.3) we see that the effect of the U(1) background connection is simply to cancel the $U(1)$ contribution, leaving us, finally, with

$$\begin{aligned} \mathcal{Z} = \frac{1}{N!} \sqrt{\frac{2\pi}{g_c^2 A_c N}} \sum_{\bar{n}_i \neq \bar{n}_j} \int_0^{2\pi} \frac{d\alpha}{\sqrt{\pi}} \exp & \left[- \left(\alpha - \frac{2\pi}{N} \sum_{i=1}^N \bar{n}_i \right)^2 - \right. \\ & \left. - 2\pi i \frac{m}{N} \left(\frac{(N-1)N}{2} - \sum_{i=1}^N \bar{n}_i \right) \right] \times \\ \times \exp \left[-\frac{g_c^2 A_c}{2} \left(\sum_{i=1}^N \bar{n}_i^2 - \frac{1}{N} \left(\sum_{i=1}^N \bar{n}_i \right)^2 \right) \right] ; & \end{aligned} \tag{3.15}$$

we will find useful to have written the partition function in eq. (3.15) as a sum over N integers, exploiting the procedure presented in [29]. It is intriguing to notice that eq. (3.15) also holds when $m_{\mathrm{nc}} \neq 0$ when N is identified with N_c given by eq. (2.9) and m with m_c given by eq. (2.18). Due to the effect of the background connection we realize that the partition function does not depend on the choice of the solution in eq. (2.11): eq. (3.15) depends on m_c only modulo N_c, which is exactly the ambiguity allowed by (2.11). Actually, for $m_{\mathrm{nc}} \neq 0$ the cancellation of the U(1) factor is not complete, but there is a surving contribution that is, however, only function of the non-commutative geometrical data and thus it is not affected by the aforementioned ambiguity. Its explicit expression is $\exp[-\frac{2\pi^2 N}{g_{\mathrm{nc}}^2 A_{\mathrm{nc}} N_c} m_{\mathrm{nc}}^2] = \exp[-\frac{2\pi^2 m_{\mathrm{nc}}^2}{g_{\mathrm{nc}}^2 A_{\mathrm{nc}}(1-\Theta m_{\mathrm{nc}})}]$. Coming back to $m_{\mathrm{nc}} = 0$, we shall use the above fact to simplify our subsequent analysis. We start choosing $|c| = 1$ since this is only a finite rescaling of θ, then we can set $m = 1$ (θ positive) or $m = -1$ (θ negative) since what we throw away is zero modulo N. We have now to perform our large-N limit in eq. (3.15): it is convenient to work in the dual representation obtained by Poisson resumming the series. According to Witten's suggestion [23], in this representation the partition function appears to be localized around the classical solutions ("instantons"), and the small area behavior is better understood within this framework.

To perform the Poisson resummation we introduce two auxiliary variables

$$
\begin{aligned}
\mathcal{Z} = & \frac{1}{N!}\sqrt{\frac{2\pi}{g_c^2 A_c N}} \sum_{\bar{n}_i \neq \bar{n}_j} \int_0^{2\pi} \frac{d\alpha}{\sqrt{\pi}} \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} \frac{d\beta}{2\pi} \times \\
& \times \exp\left[i\beta \left(y - \sum_{i=1}^{N} \bar{n}_i \right) - \left(\alpha - \frac{2\pi}{N} y \right)^2 \right] \times \\
& \times \exp\left[-2\pi i \frac{m}{N} \left(\frac{(N-1)N}{2} - y \right) - \frac{g_c^2 A_c}{2} \left(\sum_{i=1}^{N} \bar{n}_i^2 - \frac{y^2}{N} \right) \right].
\end{aligned} \tag{3.16}
$$

Next we observe that we can extend the sum over all $\bar{n}_i$ using the following trick [30]: our series is of the type

$$
\sum_{\bar{n}_i \neq \bar{n}_j} f(\bar{n}_1, \ldots, \bar{n}_N) \,, \tag{3.17}
$$

where $f(\bar{n}_1, \ldots, \bar{n}_N)$ is completely symmetric in $\bar{n}_i$. We can write

$$
\begin{aligned}
\sum_{\bar{n}_i \neq \bar{n}_j} f(\bar{n}_1, \ldots, \bar{n}_N) = & \sum_{\bar{n}_i = -\infty}^{+\infty} \sum_P (-1)^P \int_0^{2\pi} \prod_{i=1}^{N} \frac{d\theta_i}{2\pi} \exp\left[-\sum_{j=1}^{N} \theta_j \left(\bar{n}_j - \bar{n}_{P(j)} \right) \right] \times \\
& \times f(\bar{n}_1, \ldots, \bar{n}_N) \,,
\end{aligned} \tag{3.18}
$$

where no restriction appears on the $\bar{n}_i$'s. $\sum_P$ means the sum over all elements of the symmetric group S_N, $P(i)$ denotes the index i transformed by P, while $(-1)^P$ is the parity of the permutation. One recovers the original form eq. (3.6) by simply integrating over the angles θ_i and using the formula

$$
\sum_P (-1)^P \prod_{i=1}^{N} \delta_{\bar{n}_i, \bar{n}_{P(i)}} = \det\, \delta_{\bar{n}_i, \bar{n}_j} \,. \tag{3.19}
$$

The basic observation is now that, due to the symmetry of $f(\bar{n}_1, \ldots, \bar{n}_N)$, only the conjugacy classes of S_N are relevant in computing the series: to see this we use the cycle decomposition of the elements of S_N.

A conjugacy class of S_N is conveniently described by the set of non-negative integers $\{\nu_i\} = (\nu_1, \nu_2, \ldots, \nu_N)$ (we follow the description of [31]) satisfying the constraint

$$
\nu_1 + 2\nu_2 + 3\nu_3 + \cdots + N\nu_N = N \,. \tag{3.20}
$$

Every element belonging to $\{\nu_i\}$ has the same parity and can be decomposed, in a standard way, into ν_1 one-cycles, $\nu_2$two-cycles, $\ldots, \nu_N$ N-cycles. Due to the symmetry of $f(\bar{n}_1, \ldots, \bar{n}_N)$ all the elements of a conjugacy class give the same contribution in eq. (3.18), as a simple relabeling of the $\bar{n}_i$'s and θ_i's is sufficient: only the parity of the class and the number of its elements, as function of $\{\nu_i\}$, are therefore relevant to the computation of the partition function. It turns out that $(-1)^{\sum_i \nu_{2i}}$ is the parity, while the number of elements in the conjugacy class $\{\nu_i\}$ is

$$
M_{\{\nu_i\}} = \frac{N!}{1^{\nu_1}\nu_1!\, 2^{\nu_2}\nu_2! \cdots N^{\nu_N}\nu_N!} \,. \tag{3.21}
$$

The next step is to use the decomposition in cycles to perform explicitly the angular integrations: the effect is to express the full series as a finite sum of series over a decreasing number of integers. One easily realizes that a two-cycle results into the identification of two $\bar{n}_i$'s in the sum, a three-cycle into the identification of three $\bar{n}_i$'s and so on. We end up with

$$
\begin{aligned}
\mathcal{Z} = \frac{(-1)^{(N-1)m}}{N!}\sqrt{\frac{2\pi}{g_c^2 A_c N}} \sum_{\{\nu_i\}} \sum_{\bar{n}_1,\cdots\bar{n}_\nu=-\infty}^{+\infty} (-1)^{\sum_i \nu_{2i}} M_{\{\nu_i\}} \int_{-\infty}^{+\infty}\frac{d\beta}{2\pi}\, dy \int_0^{2\pi}\frac{d\alpha}{\sqrt{\pi}} \times \\
\times \exp\Bigg[-\left(\alpha - \frac{2\pi}{N}y\right)^2 + i\beta\left(y - \sum_{i_1=1}^{\nu_1}\bar{n}_{i_1} - 2\sum_{i_2=\nu_1+1}^{\nu_1+\nu_2}\bar{n}_{i_2} - 3\sum_{i_3=\nu_1+\nu_2+1}^{\nu_1+\nu_2+\nu_3}\bar{n}_{i_3} - \cdots\right) + \\
+ 2\pi i \frac{m}{N} y\Bigg] \times \\
\times \exp\left[-\frac{g_c^2 A_c}{2}\left(\sum_{i_1=1}^{\nu_1}\bar{n}_{i_1}^2 + 2\sum_{i_2=\nu_1+1}^{\nu_1+\nu_2}\bar{n}_{i_2}^2 + 3\sum_{i_3=\nu_1+\nu_2+1}^{\nu_1+\nu_2+\nu_3}\bar{n}_{i_3}^2 + \cdots\right) - \frac{g_c^2 A_c}{2}y^2\right], \qquad (3.22)
\end{aligned}
$$

where each conjugacy class has produced a sum over $\nu = \nu_1 + \nu_2 + \cdots + \nu_N$ integers; of course if some ν_j is zero, the integers $n_{\nu_1+\cdots\nu_{j-1}+1}, \ldots, n_{\nu_1+\cdots\nu_{j-1}+\nu_j}$ do not appear. The Poisson resummation is, at this point, almost trivial, being the set $(n_1, \ldots, n_\nu)$ unrestricted: it requires in our case only gaussian integrations. The remaining integrals over the auxiliary variables are also easily done. The final result, expressing the original partition function as a sum over "dual" integers m_i's, is:

$$
\begin{aligned}
\mathcal{Z} = \frac{(-1)^{(N-1)m}}{N}\sqrt{\frac{2\pi}{g_c^2 A_c N}} \times \\
\times \Bigg[1 + {\sum_{\{\nu_i\}}}' (-1)^{\sum_i \nu_{2i}} Z_{\{\nu_i\}} \left(\frac{2\pi}{g_c^2 A_c}\right)^{\frac{\nu-1}{2}} \sum_{m_1,\ldots,m_\nu=-\infty}^{+\infty} \delta\left(m - \sum_{i=1}^{\nu} m_i\right) \times \\
\times \exp\Bigg[-\frac{2\pi^2}{g_c^2 A_c}\Bigg(\sum_{i_1=1}^{\nu_1}\left(m_{i_1} - \frac{m}{N}\right)^2 + \frac{1}{2}\sum_{i_2=\nu_1+1}^{\nu_1+\nu_2}\left(m_{i_2} - 2\frac{m}{N}\right)^2 + \\
+ \frac{1}{3}\sum_{i_3=\nu_1+\nu_2+1}^{\nu_1+\nu_2+\nu_3}\left(m_{i_2} - 3\frac{m}{N}\right)^2 + \\
+ \cdots \frac{1}{N-1}\sum_{i_{N-1}}^{\nu}\left(m_{i_{N-1}} - (N-1)\frac{m}{N}\right)^2\Bigg)\Bigg]\Bigg], \qquad (3.23)
\end{aligned}
$$

where the prime means that we are summing over all the partitions of N except the identity, $\nu_1 + 2\nu_2 + \cdots + (N-1)\nu_{N-1} = N$ and

$$
Z_{\{\nu_i\}} = \frac{N^{3/2}\left[1^{\nu_1} 2^{\nu_2} 3^{\nu_3} \cdots (N-1)^{\nu_{N-1}}\right]^{-3/2}}{\nu_1! \nu_2! \nu_3! \cdots \nu_{N-1}!}. \qquad (3.24)
$$

These formulae have nice interpretation in term of instantons: the partition function appears to be localized around its classical solutions [23] and the limit $g_c^2 A_c \to 0$ is related to the properties of the moduli space of flat connections. Using eqs. (2.16), (2.17) and (2.18) we can rewrite the sum in eq. (3.23) in the following way (we consider the case of θ positive):

$$\sum_{\{\nu_i\}}{}^{'}(-1)^{\sum_i \nu_{2i}} Z_{\{\nu_i\}} \left(\frac{N^2}{g_{\rm nc}^2\theta}\right)^{\frac{\nu-1}{2}} \sum_{m_1,\cdots m_\nu=-\infty}^{+\infty} \delta\left(\sum_{i=1}^{\nu} m_i+1\right)\times$$
$$\times \exp\left[-\frac{\pi N^2}{g_{\rm nc}^2\theta}\left(\sum_{i_1=1}^{\nu_1}\left(m_{i_1}+\frac{1}{N}\right)^2+\frac{1}{2}\sum_{i_2=\nu_1+1}^{\nu_1+\nu_2}\left(m_{i_2}+\frac{2}{N}\right)^2+\right.\right. \tag{3.25}$$
$$\left.\left.+\frac{1}{3}\sum_{i_3=\nu_1+\nu_2+1}^{\nu_1+\nu_2+\nu_3}\left(m_{i_3}+\frac{3}{N}\right)^2+\cdots\frac{1}{N-1}\sum_{i_{N-1}}^{\nu}\left(m_{i_{N-1}}+\frac{N-1}{N}\right)^2\right)\right].$$

At this point we have to perform our large-N limit: in so doing we are recovering the theory on the non-commutative plane. We will neglect exponentially suppressed contribution: in this way we expect to find the partition function expressed through a semiclassical expansion, taking into account all the finite action classical solutions. We do not consider here the (possible) effect of non-finite action contributions.

The problem is therefore reduced to find all the configurations that are not exponentially suppressed in the limit $N\to\infty$: we have to single out both the partitions of N and the instanton numbers m_i. A very simple set surviving in the limit is the following: let us consider the conjugacy classes composed by a single cycle of order N and the other ones finite

$$\nu_{N-k}=1\,;\qquad \nu_1+2\nu_2+\cdots+k\nu_k=k\,, \tag{3.26}$$

where k is some positive integer. The choice $m_{i_1}=m_{i_2}=\cdots=m_{i_k}=0$; $m_{i_{N-k}}=-1$ for the associated instanton numbers leads to a finite exponent

$$\frac{\pi N^2}{g_{\rm nc}^2\theta}\left(\frac{\nu_1+2\nu_2+\cdots+k\nu_k}{N^2}+\frac{k^2}{N-k}\frac{1}{N^2}\right)=\frac{\pi k}{g_{\rm nc}^2\theta}+{\rm O}\left(\frac{1}{N}\right). \tag{3.27}$$

We claim that these are the dominant configurations: we do not have a rigorous proof but it is possible to show, first of all, that configurations with no cycle of order N are suppressed. Suppose to have a partition with cycles up to k

$$\nu_1+2\nu_2+\cdots+k\nu_k=N\,.$$

Computing the exponent and using the fact that $\sum_{i=1}^{\nu} m_i=-1$ we obtain:

$$\frac{\pi N^2}{g_{\rm nc}^2\theta}\left(\sum_{i_1} m_{i_1}^2+\frac{1}{2}\sum_{i_2} m_{i_2}^2+\cdots+\frac{1}{k}\sum_{i_k} m_{i_k}^2-\frac{1}{N}\right); \tag{3.28}$$

the contribution inside the parenthesis is O(1). On the other hand when two cycles are of order N we get a suppressed configuration as well. Let us consider for example the partition $N_1+N_2=N$ ($N_1<N_2$): because the ratios $N_1/N, N_2/N$ are finite in the limit, we can minimize the action along $m_1+m_2=-1$ finding

$$\frac{\pi N^2}{g_{\rm nc}^2\theta}\left(\frac{1}{N_1}\left(m_1+\frac{N_1}{N}\right)^2+\frac{1}{N_2}\left(m_2+\frac{N_2}{N}\right)^2\right)=\frac{\pi N_1}{g_{\rm nc}^2\theta}\left(1+\frac{N_1}{N_2}\right).$$

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It is easy to generalize the argument when finite-size cycles are also present and more than two cycles of size N appear in the decomposition. We assume therefore that the configurations of eq. (3.26) are the dominant ones in the large-N limit: summing over k we obtain

$$\mathcal{Z} = \frac{1}{N\sqrt{g_{\rm nc}^2\theta}}\sum_{k=0}^{+\infty}(-1)^k \exp\left(-\frac{\pi k}{g_{\rm nc}^2\theta}\right)\sum_{\{\nu_i\}_k}\frac{(-1)^{\sum_i \nu_{2i}}}{[2^{\nu_2}\cdots k^{\nu_k}]^{3/2}}\frac{1}{\nu_1!\cdots\nu_k!}\left(\frac{A_{\rm nc}}{\sqrt{2\pi g_{\rm nc}^2\theta^3}}\right)^{\nu}, \quad (3.29)$$

where we have substituted N with the area of the non-commutative plane and the explicit form of $Z_{\{\nu_i\}}$ has been taken into account. This formula displays a certain number of remarkable features: probably the most interesting is that the partition function is expressed as a sum over "fluxons". In [18, 19] it has been shown that pure U(1) gauge theory on the non-commutative plane admits finite energy instanton solutions carrying quantized magnetic flux. A peculiar feature is that fluxons exist with only one sign of the magnetic charge, B being aligned with θ. The classical action S is also peculiar being linear in the magnetic number: it turns out that for a fluxon solution of charge m we have

$$S = \frac{\pi m}{g^2\theta}\,. \quad (3.30)$$

Consistently with the non existence of classical solutions in the commutative case, we see that this action is singular as $\theta \to 0$. Equation (3.29) reproduces correctly the sum over the classical action: one is therefore tempted to interpret the coefficient associated to the charge k sector as the effect of the fluctuation around the fluxon solution. According to [18, 19], a fluxon of charge k carries a $2k$-dimensional moduli space reflecting the centers of the k elementary vortices of which is made. Moreover these solutions have been proved to be unstable [19, 32] and a sensible semiclassical expansion would seem therefore to be hopeless. On the other hand it is well known that the partition function of ordinary YM_2, on compact surface, is localized around its critical points [23], due to a generalization of the Duistermaat-Heckman formula: the path integral is exact in the semiclassical approximation although the classical solutions are unstable. In the non-commutative case, at least in the limit of large non-commutative area, the very same phenomenon seems to occur: the coefficient associated to k has the following appealing explanation in terms of moduli. If we assume that an elementary vortex could carry *any* integer charge, the sum over the partition of k in eq. (3.29) has a natural interpretation: a magnetic charge k appears to be composed by ν_1 elementary vortex of charge 1, ν_2 of charge 2 and so on. Vortices of equal charge, inside the fluxon, are identical, therefore the factor $\frac{1}{\nu_1!\nu_2!\cdots\nu_k!}$ appears. The integration over the positions is also correctly reproduced, the area factor appearing with exponent $\nu_1+\nu_2\cdots+\nu_k$, that represents the total number of elementary constituents. The other factors have to be related to the computation of the quantum fluctuations around the fluxons. The solutions found in [18, 19], in this picture, are the one related to $\{\nu_1 = k,\ \nu_2 = \cdots = \nu_k = 0\}$. It could be that on compact space a larger set of solutions (we remark that we are decompactifying a torus) is present but we do not have an answer at this moment. Alternatively one could write the partition function as

$$\mathcal{Z} = \sum_{k=0}^{+\infty}\exp\left(-\frac{\pi k}{g_{\rm nc}^2\theta}\right)\frac{a_k(A_{\rm nc}/\sqrt{g_{\rm nc}\theta})}{k!}\,,$$

and to understand a_k as the total result of the fluctuations around the classical solution. In any case eq. (3.29) may provide an answer to the question posed in ref. [19] about the possibility to compute the non-commutative partition function (the regulator we use is essentially the area of torus) in the semiclassical approximation. At this point we can actually go further: it is possible to resum exactly the series in eq. (3.29). To compute the sum we observe that the constraint $\nu_1 + 2\nu_2 + \cdots + k\nu_k = k$ can be implemented by introducing an angular variable to obtain

$$\begin{aligned}\mathcal{Z} &= \sum_{k=0}^{+\infty}(-1)^k \exp\left(-\frac{\pi k}{g_{\rm nc}^2\theta}\right) \sum_{\{\nu_i\}=0}^{+\infty} \int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left[i(\nu_1 + 2\nu_2 + \cdots + k\nu_k - k)\theta\right] \times \\ &\times \left(\frac{A_{\rm nc}}{\sqrt{2\pi g_{\rm nc}^2\theta^3}}\right)^{\nu_1+\nu_2+\cdots+\nu_k} (-1)^{\nu_2+\nu_4+\cdots} \frac{(1^{-3/2})^{\nu_1}}{\nu_1!} \frac{(2^{-3/2})^{\nu_2}}{\nu_2!} \cdots \frac{(N^{-3/2})^{\nu_k}}{\nu_k!}, \end{aligned} \tag{3.31}$$

where we disregarded the overall multiplicative constant. The sum over ν_i's is now simple, giving

$$\mathcal{Z} = \sum_{k=0}^{+\infty}(-1)^k \exp\left(-\frac{\pi k}{g_{\rm nc}^2\theta}\right) \int_0^{2\pi} \frac{d\theta}{2\pi} \mathrm{e}^{-ik\theta} \exp\left[-\frac{A_{\rm nc}}{\sqrt{2\pi g_{\rm nc}^2\theta^3}} \sum_{l=1}^{k} \frac{\mathrm{e}^{il\theta}(-1)^l}{l^{3/2}}\right], \tag{3.32}$$

that can be expressed as a contour integral in the complex plane

$$\mathcal{Z} = \sum_{k=0}^{+\infty}(-1)^k \exp\left(-\frac{\pi k}{g_{\rm nc}^2\theta}\right) \frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{dz}{z^{k+1}} \exp\left[z\Phi\left(-z;\frac{3}{2};1\right) \frac{A_{\rm nc}}{\sqrt{2\pi g_{\rm nc}^2\theta^3}}\right], \tag{3.33}$$

where $\mathcal{C}_0$ rounds the origin anticlockwise, sufficiently close so that the function $\Phi(z;s;\mu)$

$$\Phi(z;s;\mu) = \sum_{k=0}^{+\infty} \frac{z^k}{(k+\mu)^s}; \tag{3.34}$$

is analytic ($|z| < 1$ to avoid the cut from $z = 1$ to ∞ of the Φ function). Now $\mathcal{C}_0$ can be chosen so that $|\frac{e^{-\pi/g_{\rm nc}^2\theta}}{z}| < 1$ and the series in k is easily done

$$\mathcal{Z} = \frac{1}{2\pi i} \int_{\mathcal{C}_0} \frac{dz}{z + e^{-\pi/g_{\rm nc}^2\theta}} \exp\left[z\Phi\left(-z;\frac{3}{2};1\right) \frac{A_{\rm nc}}{\sqrt{2\pi g_{\rm nc}^2\theta^3}}\right], \tag{3.35}$$

leading to

$$\mathcal{Z} = \exp\left[-\frac{e^{-\pi/g_{\rm nc}^2\theta}}{\sqrt{2\pi g_{\rm nc}^2\theta^3}} \Phi\left(e^{-\pi/g_{\rm nc}^2\theta};\frac{3}{2};1\right) A_{\rm nc}\right]. \tag{3.36}$$

The formula is interesting: the partition function, in our approximation, appears to be extensive and the whole instantons series has been resummed into the coefficient of $A_{\rm nc}$ in eq. (3.36). This is typical of the diluite instanton gas picture, in which instantons are taken to be not interacting. In our case this result is exact, fluxons being non-interacting objects [19]. We will use the expression of $\mathcal{Z}$ as normalization for the correlation function of two open Wilson lines.

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4. The correlator of open Wilson Lines

Now we would like to apply our machinery to the computation of some physical observables: in a non-commutative gauge theory the basic gauge invariant quantities are the correlators of open Wilson lines [33] (see also [34], where they were originally proposed, and [35] concerning their supergravity description). Pure non-commutative gauge theory has no local gauge invariant operators: it is instead possible to construct a gauge invariant observable out of the open Wilson lines provided they have a transverse momentum (we will consider for simplicity straight open Wilson lines)

$$W(\vec{p}) = \int d^2x\, \Omega(x_i \to x_i + \theta_{ij}p_j) \star \exp(i\vec{p}\,\vec{x})\,, \tag{4.1}$$

where $\Omega(x \to y)$ is an open Wilson line, path ordered with respect to the non-commutative star product, and stretching between the point x and y in the non-commutative plane. Because of the noncommutativity, the term $\exp(i\vec{p}\,\vec{x})$ is a translation operator in the direction transverse to $\vec{p}$ which for gauge invariance must relate the two endpoints of the open Wilson line [33]. Quantum correlators were also considered in [33, 36], exhibiting an interesting string-like behavior. We are interested here in computing the simplest non-trivial correlator (the two-point function) for straight, parallel Wilson lines. We will obtain a finite result, in the limit of long lines, expressed through an expansion around the classical solutions of the theory, in agreement with the result for the partition function.

We start by considering the following observable, stretching along x_2, on the non-commutative torus:

$$\begin{aligned} W(k_1, n_2) = \frac{1}{4\pi^2 R_{\rm nc}^2}\int_0^{2\pi R_{\rm nc}} & d^2x\, \Omega\left(x_1; x_2 \to x_2 + 2\pi R_{\rm nc} n_2 - 2\pi R_{\rm nc}\frac{c}{N}k_1\right) \star \\ & \star \exp\left(i\frac{k_1 x}{R_{\rm nc}}\right). \end{aligned} \tag{4.2}$$

This is the direct generalization of eq. (4.1): n_2 describes the number of winding along the x_2 coordinate and k_1 is the integer associated to the transverse momentum $k_1/R_{\rm nc}$. In order to count into n_2 all the windings we take $|ck_1| < N$. The normalization of the line is chosen so that $W(0,0) = 1$. Let us consider the decompactification limit in eq. (4.2) (necessarily implying $n_2 = 0$), requiring to have a finite transverse momentum, $\frac{k_1}{R_{\rm nc}} \to p$ as $R_{\rm nc} \to \infty$. Taking into account eq. (2.15) we see that k_1 has to be scaled as

$$k_1 = \sqrt{\frac{N|\theta|}{2\pi}}\,p\,. \tag{4.3}$$

In the decompactification limit the total length L of the Wilson line turns out to be finite

$$L = \frac{2\pi R_{\rm nc} k_1}{N} = p|\theta|\,, \tag{4.4}$$

with the only restriction $p\sqrt{\theta/2\pi} < \sqrt{N}$ (that is the original no winding request).

The next step is to write down explicitly the Morita transformed operator: the mapping has been derived in [14, 37, 38], showing that open non-commutative Wilson lines map into the Polyakov loops of ordinary Yang-Mills theory,

$$\begin{aligned} W(k_1,0) &= W^{(k_1)}\,, \\ W^{(k_1)} &= \frac{1}{4\pi^2 R_c^2}\int_0^{2\pi R_c} d^2x\,\frac{1}{N}\,\mathrm{Tr}\left[\Omega^{(k_1)}(x_1)\right]. \end{aligned} \tag{4.5}$$

$\Omega^{(k_1)}(x_1)$ is the holonomy derived from a closed path, winding k_1 times along the x_2 direction, the trace in eq. (4.5) has to be taken in the fundamental representation of U(N). We see therefore that under Morita equivalence the computation of open Wilson lines correlators has been mapped in an analogous problem for conventional Polyakov loops. As for the classical action, we have to consider in the definition of $\Omega^{(k_1)}(x_1)$ the contribution of the background connection eq. (2.12): it turns out that the holonomy of the fixed abelian background has to be subtracted, leaving us with the computation of $\Omega^{(k_1)}(x_1)$ for $U(1)\otimes SU(N)/\mathbb{Z}_N$ (in the flux sector m_c), where the U(1) contribution is taken in the trivial sector. Let us consider the simplest correlation function, involving just two open Wilson lines [33],

$$W_2(k_1) = \langle W(k_1,0)W(-k_1,0)\rangle : \tag{4.6}$$

under Morita equivalence we have

$$W_2(k_1) = \frac{1}{4\pi^2 R_c^2}\int_0^{2\pi R_c} dx_1 dy_1 \left\langle \frac{1}{N}\,\mathrm{Tr}\left[\Omega^{(k_1)}(x_1)\right]\frac{1}{N}\,\mathrm{Tr}\left[\Omega^{(-k_1)}(y_1)\right]\right\rangle, \tag{4.7}$$

where the integrations over x_2, y_2 lead simply to a volume factor because the correlator does not depend on them. We can factorize another volume factor by noticing that due to translational invariance the correlation function only depends on the relative coordinate and distinguishing the physically inequivalent configurations we arrive to

$$W_2(k_1) = \frac{1}{2\pi R_c}\left[\int_0^{\pi R_c} dx \left\langle \frac{1}{N}\,\mathrm{Tr}\left[\Omega^{(k_1)}(x)\right]\frac{1}{N}\,\mathrm{Tr}\left[\Omega^{(-k_1)}(0)\right]\right\rangle + k_1 \to -k_1\right]. \tag{4.8}$$

The correlation function has to be computed in the m-th 't Hooft sector (we will consider at the end $m=\pm 1$ according to the sign of θ as we have done for the partition function) and we can again take advantage of the Migdal-Rusakov's formulae for U(N):

$$\begin{aligned} &\frac{1}{N^2}\left\langle \mathrm{Tr}\left[\Omega^{(k_1)}(x)\right]\,\mathrm{Tr}\left[\Omega^{(-k_1)}(0)\right]\right\rangle = \\ &= \frac{1}{\mathcal{Z}}\sum_{R,S}\exp\left[-\frac{g_c^2(A_c-A_2)}{2}C_2(R)\right]\exp\left[-\frac{g_c^2 A_2}{2}C_2(S)\right]\times \\ &\qquad \times\int dU_1\,\mathcal{X}_R(U_1)\mathcal{X}_F(U_1^{k_1})\mathcal{X}_S^\dagger(U_1)\int dU_2\,\mathcal{X}_S(U_2)\mathcal{X}_F(U_2^{-k_1})\mathcal{X}_R^\dagger(U_2)\,, \end{aligned} \tag{4.9}$$

A_2 being the total area contained between the two loops ($A_2 = 2\pi R_c x$) and $\mathcal{X}_F$ is the character of the fundamental representation of U(N). It is not difficult single out the relevant $\mathrm{U}(1)\otimes\mathrm{SU}(N)/\mathbb{Z}_N$ contribution, along the same lines of the previous chapter. In term of the $\bar{n}_i$'s we have the analogue of eq. (3.15)

$$\begin{aligned}
&\frac{1}{N^2}\left\langle \mathrm{Tr}\left[\Omega^{(k_1)}(x)\right]\,\mathrm{Tr}\left[\Omega^{(-k_1)}(0)\right]\right\rangle =\\
&=\frac{1}{\mathcal{Z}N!}\sqrt{\frac{2\pi}{g_c^2A_cN}}\exp\left[\frac{g_c^2A_2^2k_1^2}{2A_cN}-\frac{g_c^2A_2k_1^2}{2}\right]\sum_{\bar{n}_i\neq\bar{n}_j}\int_0^{2\pi}\frac{d\alpha}{\sqrt{\pi}}\times\\
&\quad\times\exp\left[-\left(\alpha-\frac{2\pi}{N}\sum_{i=1}^N\bar{n}_i\right)^2-2\pi i\frac{m}{N}\left(\frac{(N-1)N}{2}-\sum_{i=1}^N\bar{n}_i\right)\right]\times\\
&\quad\times\exp\left[-\frac{g_c^2A_c}{2}\left(\sum_{i=1}^N\bar{n}_i^2-\frac{1}{N}\left(\sum_{i=1}^N\bar{n}_i\right)^2\right)\right]\frac{1}{N}\sum_{j=1}^N\exp\left[-g_c^2A_2k_1\left(\bar{n}_j-\frac{1}{N}\sum_{i=1}^N\bar{n}_i\right)\right].
\end{aligned}\tag{4.10}$$

We will take for simplicity, from now on, N odd. We introduce the cycles decomposition, as in eq. (3.22), and, using the auxiliary integration variables, we can write the sum in eq. (4.10) as

$$\begin{aligned}
&\frac{1}{N}\sum_{\{\nu_i\}}\sum_{\bar{n}_1,\cdots\bar{n}_\nu=-\infty}^{+\infty}(-1)^{\sum_i\nu_{2i}}M_{\{\nu_i\}}\int_{-\infty}^{+\infty}\frac{d\beta}{2\pi}\,dy\int_0^{2\pi}\frac{d\alpha}{\sqrt{\pi}}\exp\left[-\left(\alpha-\frac{2\pi}{N}y\right)^2\right]\times\\
&\times\exp\left[-\frac{g_c^2A_c}{2}\left(\sum_{i_1=1}^{\nu_1}\bar{n}_{i_1}^2+2\sum_{i_2=\nu_1+1}^{\nu_1+\nu_2}\bar{n}_{i_2}^2+3\sum_{i_3=\nu_1+\nu_2+1}^{\nu_1+\nu_2+\nu_3}\bar{n}_{i_3}^2+\cdots\right)-\frac{g_c^2A_c}{2}y^2\right]\times\\
&\times\exp\left[i\beta\left(y-\sum_{i_1=1}^{\nu_1}\bar{n}_{i_1}-2\sum_{i_2=\nu_1+1}^{\nu_1+\nu_2}\bar{n}_{i_2}-3\sum_{i_3=\nu_1+\nu_2+1}^{\nu_1+\nu_2+\nu_3}\bar{n}_{i_3}-\cdots\right)+\right.\\
&\qquad\left.+\frac{2\pi iy}{N}\left(m-i\frac{g_c^2A_2k_1}{2\pi}\right)\right]\left[\nu_1\exp\left(-g_c^2A_2k_1n_1\right)+2\nu_2\exp\left(-g_c^2A_2k_1n_{\nu_1+1}\right)+\cdots\right],
\end{aligned}\tag{4.11}$$

where we have explicitly taken into account the symmetry between the integers associated to cycles of equal length. The Poisson resummation is now simple because we can use our previous result realizing that we have the shifts

$$\begin{aligned}
&m\longrightarrow m-i\frac{g_c^2A_2k_1}{2\pi},\\
&m_1\longrightarrow m_1-i\frac{g_c^2A_2k_1}{2\pi},\qquad m_{\nu_1+1}\longrightarrow m_{\nu_1+1}-i\frac{g_c^2A_2k_1}{2\pi},\ \cdots.
\end{aligned}\tag{4.12}$$

The final result can be written as

$$\begin{aligned}
&\frac{1}{N^2}\left\langle \mathrm{Tr}\left[\Omega^{(k_1)}(x)\right]\,\mathrm{Tr}\left[\Omega^{(-k_1)}(0)\right]\right\rangle =\\
&=\frac{1}{\mathcal{Z}N}\sqrt{\frac{2\pi}{g_c^2A_cN}}\exp\left[\frac{g_c^2A_2^2k_1^2}{2A_cN}-\frac{g_c^2A_2k_1^2}{2}\right]\times\\
&\quad\times\left\{1+{\sum_{\{\nu_i\}}}'(-1)^{\sum_i\nu_{2i}}Z_{\{\nu_i\}}\left(\frac{2\pi}{g_c^2A_c}\right)^{\frac{\nu-1}{2}}\sum_{m_1,\cdots m_\nu=-\infty}^{+\infty}\delta\left(m-\sum_{i=1}^{\nu}m_i\right)\times\right.
\end{aligned}$$

$$\times \exp\left[-\frac{2\pi^2}{g_c^2 A_c}\left(\sum_{i_1=1}^{\nu_1}\left(m_{i_1}-\frac{m}{N}\right)^2+\ +\frac{1}{2}\sum_{i_2=\nu_1+1}^{\nu_1+\nu_2}\left(m_{i_2}-2\frac{m}{N}\right)^2+\cdots\right)\right]\times$$
$$\times\frac{1}{N}\sum_{l=1}^{N-1} l\nu_l \exp\left[\frac{g_c^2 A_2^2 k_1^2}{2A_c}\left(\frac{1}{l}+\frac{l}{N^2}-\frac{2}{N}\right)+\right.$$
$$\left.\left.+2\pi i\frac{A_2 k_1}{A_c}\left(\frac{1}{l}-\frac{1}{N}\right)\left(m_{\nu_1+\cdots+\nu_{l-1}+1}-\frac{ml}{N}\right)\right]\right\}. \quad (4.13)$$

Next we have to evaluate the above expression on the configurations found in the previous section and to take the large-N limit: we arrive to the following expression (we write the sum in terms of the non-commutative data)

$$1+\frac{1}{\mathcal{Z}N\sqrt{g_{\rm nc}^2\theta}}\sum_{k=1}^{+\infty}(-1)^k\exp\left(-\frac{\pi k}{g_{\rm nc}^2\theta}\right)\sum_{\{\nu_k\}}{}^{'}(-1)^{\sum_i \nu_{2i}}Z_{\{\nu_k\}}\left(\frac{A_{\rm nc}}{\sqrt{2\pi g_{\rm nc}^2\theta^3}}\right)^{\nu}\times$$
$$\times\frac{1}{N}\left(\sum_{l=1}^{k} l\nu_l\exp\left[i\frac{xp}{2\pi}+\frac{1}{l}\frac{g_{\rm nc}^2p^2x^2\theta}{16\pi^3}\right]-k\right). \quad (4.14)$$

In order to evaluate the sum let us discuss

$$F(xp,y)=\sum_{k=1}^{+\infty}(-1)^k\exp\left(-\frac{\pi k}{g_{\rm nc}^2\theta}\right)\sum_{\{\nu_k\}}{}^{'}(-1)^{\sum_i \nu_{2i}}Z_{\{\nu_k\}}y^{\nu}\frac{1}{N}\sum_{l=1}^{k}l\nu_l f_l(xp),$$

where $y=A_{\rm nc}/\sqrt{2\pi g_{\rm nc}^2\theta^3}$ and $f_l(xp)=\exp[i\frac{xp}{2\pi}+\frac{1}{l}\frac{g_{\rm nc}^2p^2x^2\theta}{16\pi^3}]$: the correlator is now simply given by

$$\frac{1}{N^2}\left\langle {\rm Tr}\left[\Omega^{(k_1)}(x)\right]\,{\rm Tr}\left[\Omega^{(-k_1)}(0)\right]\right\rangle=1+\frac{1}{\mathcal{Z}N\sqrt{g_{\rm nc}^2\theta}}\left[F(xp,y)-F(0,y)\right].$$

Using the contour representation in eq. (3.33) we have

$$F(xp,y)=\sum_{k=1}^{+\infty}(-1)^k\exp\left(-\frac{\pi k}{g_{\rm nc}^2\theta}\right)\frac{y}{N}\sum_{l=1}^{k}f_l(xp)\frac{(-1)^l}{\sqrt{l}}\frac{1}{2\pi i}\int_{\mathcal{C}_0}\frac{dz}{z^{k+1}}z^l\times$$
$$\times\exp\left[z\Phi\left(-z;\frac{3}{2};1\right)y\right] \quad (4.15)$$
$$=\sum_{k=0}^{+\infty}(-1)^k\exp\left(-\frac{\pi k}{g_{\rm nc}^2\theta}\right)\frac{y}{N}\sum_{l=1}^{+\infty}f_l(xp)\frac{(-1)^l}{\sqrt{l}}\frac{1}{2\pi i}\int_{\mathcal{C}_0}\frac{dz}{z^{k+1}}z^l\exp\left[z\Phi\left(-z;\frac{3}{2};1\right)y\right].$$

Performing the geometric series in k and evaluating the contour integrals on the pole we obtain

$$F(xp,y)=-\mathcal{Z}N\sqrt{2\pi}\sum_{l=1}^{+\infty}f_l(xp)\frac{\exp(-\pi l/g_{\rm nc}^2\theta)}{\sqrt{l}}. \quad (4.16)$$

The partition function has been explicitly factorized out leaving us with the result

$$\frac{1}{N^2}\left\langle {\rm Tr}\left[\Omega^{(k_1)}(x)\right]\,{\rm Tr}\left[\Omega^{(-k_1)}(0)\right]\right\rangle=1+\sqrt{\frac{2\pi}{g_{\rm nc}^2\theta}}\sum_{l=1}^{+\infty}\frac{\exp(-\pi l/g_{\rm nc}^2\theta)}{\sqrt{l}}\left(1-f_l(xp)\right). \quad (4.17)$$

Let us notice that because $f_l(0) = 1$ we still have that for vanishing length the correlator is normalized to one. In terms of the length of the lines the result reads

$$1+\sqrt{\frac{2\pi}{g_{\rm nc}^2\theta}}\sum_{l=1}^{+\infty}\frac{\exp(-\pi l/g_{\rm nc}^2\theta)}{\sqrt{l}}\left(1-\exp\left[i\frac{Lx}{\theta}+\frac{1}{l}\frac{g_{\rm nc}^2L^2x^2}{16\pi^3\theta}\right]\right). \tag{4.18}$$

This is the unintegrated expression: it seems still to have an interpretation in terms of fluxons. The phase is reminiscent of the result of [19], where the classical Wilson loop, evaluated on fluxons, has exactly the same form (Lx is the area determined by lines), not depending on the coupling constant $g_{\rm nc}^2$ nor on the fluxon charge. The other exponential term has a peculiar dependence $1/l$ that calls for an explanation: it would be naturally identified as the contribution of the fluctuations around the classical fluxon and it would be important to see if such a strange dependence could appear directly from some computations on the non-commutative plane. The physical observable, nevertheless, is the integral of eq. (4.17) that, because $R_c \to 0$ as $1/\sqrt{N}$, seems to go to 1 (see eq. (4.8)). Actually we can obtain a non-trivial result, in our approximation, considering Wilson lines that are very long, or, if you want, with very high momentum p. Coming back to the definition of L in eq. (4.4), we see that we can consistently consider Wilson lines of length

$$L=\frac{\lambda}{\pi}R_{\rm nc}\,,$$

with arbitrary $0 < \lambda < 1/2$: taking into account the scaling of L we obtain the correlator

$$W_2(\lambda)=1+\frac{1}{2}\int_{-1}^{1}dz\sqrt{\frac{2\pi}{g_{\rm nc}^2\theta}}\sum_{l=1}^{+\infty}\frac{\exp(-\pi l/g_{\rm nc}^2\theta)}{\sqrt{l}}\left(1-\exp\left[i\lambda z+\frac{1}{l}\frac{g_{\rm nc}^2\theta\lambda^2z^2}{4\pi}\right]\right). \tag{4.19}$$

The fact that only for long Wilson lines our computation leads to a non-trivial result may be related to point-like character of the coupling between Wilson lines and fluxons (see the discussion in [19]) and therefore only when the line is enough long the interaction becomes important. In any case we do not have a satisfactory argument to support this thesis and it might happen that non-parallel Wilson lines be non trivial or one had to compute higher order correlators.

5. Conclusions

We have explored the possibility to study dynamical properties of non-commutative gauge theories using the powerful tool of Morita equivalence. We have restricted our attention to the two-dimensional case, being the simplest situation in which concrete computations can be performed. When formulated on a two-torus with a rational non-commutativity parameter θ, the U(1) gauge theory maps, under a Morita transformation, into a usual U(N) theory in a given t' Hooft sector. We have shown that the non-commutative theory on the plane with arbitrary θ parameter can be obtained by a suitable decompactification limit, involving a series of rational approximants. Morita equivalence translates this procedure into a non-standard large-N limit in the (dual) commutative U(N) theory: as N goes to infinity not only the coupling constant scales with $1/N$ but also the commutative torus shrinks to zero-size in the same way. The t'Hooft flux has to be taken fixed. Starting from the exact

Migdal-Rusakov solution for Yang-Mills theory on the torus, we have been able to perform such a limit on the partition function, by going to a dual representation obtained from the Poisson resummation of the original series over the Young tableaux integers. We have seen that there are finite action configurations surviving in the limit (being not exponentially suppressed as N goes to infinity) and the partition function appears therefore localized around them. This is probably our most interesting result: these finite action configurations are in correspondence with the classical solutions found by Polychronakos and by Gross and Nekrasov on the non-commutative plane [18, 19] and the partition function we obtain has a precise interpretation as a semiclassical expansion around them. The whole series can actually be resummed, leading to a result that is extensive in the area: our expression closely resembles a dilute instanton gas approximation, the fluxons being indeed not interacting. We have then shown how to compute Wilson lines correlators: we have carried out the simple case of two parallel Wilson lines. A non-trivial result, having an instanton interpretation, has been obtained in the long lines-high momentum limit. We think that our computations are a first step towards the possibility to solve completely the theory, on the non-commutative plane, having reduced the problem, in principle, to a sort of large-N small area limit of the usual Yang-Mills theory on a torus. Many aspects remain to be explored: first of all we have assumed that the resulting theory on the plane does not depend on the particular series of rational approximants, a fact that has to be checked (see [15, 14] for a discussion at finite volume). While the generalization to a non-abelian non-commutative theory seems not difficult (we have studied the U(1) case with vanishing non-commutative Chern class), it remains open the possibility to consider general fluxes on the non-commutative side and then to sum over them, possibly with some instanton angle. Moreover we have not considered, in the spirit of the semiclassical approximation, the possibility that infinite action configurations might be relevant on the plane, once resummed. On the other hand, Witten has shown that ordinary Yang-Mills theory on compact surface is localized around its classical solutions, therefore our result could be something more than a semiclassical approximation. It would be very important, in this sense, to have some computations (hopefully non perturbative) directly on the non-commutative plane for Wilson line correlators and to check it against the calculation done along our procedure. The relation between our finite action configurations and the exact solitons on non-commutative tori, found in [39, 40, 41] may also shed some light on the mathematical structure beyond the limit.

Finally closed Wilson loop could also be studied: in [20] a perturbative computation has shown that interesting features concerning smoothness in θ and large-θ limit can be drawn from Wilson loops analysis. We will present, in a forthcoming paper [24], the result of our investigations on Wilson loops and how the procedure we have presented here is consistent, in a particular limit, with the resummation of the perturbative series on the non-commutative plane.

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